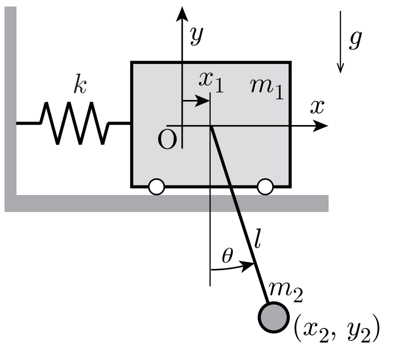
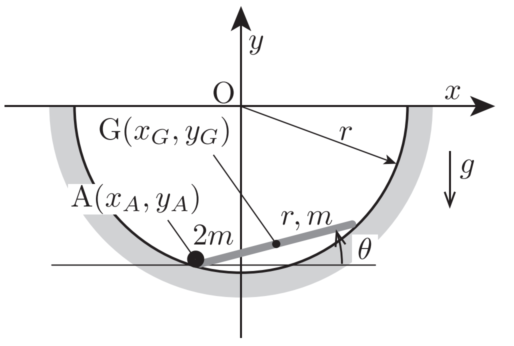
Question I. As shown in the right figure, a carriage (mass m_1) is connected to a wall with a linear spring (spring constant k). A pendulum consisting of a mass point m_2 and a massless rigid beam (length l) is connected to the carriage. A horizontal displacement of the carriage, x_1, is determined with reference to the static equilibrium position and an angular displacement of the pendulum, θ, is determined as an angle from the vertical axis. Answer the following questions. The gravitational acceleration is g as indicated by an arrow in the figure. (25 points)



(1) Coordinates of a mass point of the pendulum is (x_2, y_2). Find the displacement (x_2, y_2) and the velocity $(\dot{x}_2, \dot{y}_2)$ of the pendulum using x_1, θ, $\dot{x}_1$, and $\dot{\theta}$.
(2) Determine the kinetic energy T and the potential energy U of this system.
(3) Derive the Lagrange's equations of motion of the system using x_1 and θ as the generalized coordinates.
(4) Linearize the equations of motion and express them in matrix form $\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}$ when $|\theta|$ and $|\dot{\theta}|$ are small. Here $\boldsymbol{M}$ and $\boldsymbol{K}$ denote the mass and stiffness matrices respectively, $\boldsymbol{x} = [x_1 \;\; \theta]^T$ is generalized displacement vector and $[\;]^T$ means transpose.
(5) Find the characteristic equation (frequency equation) of this linearized system when $k = 0$, and find the eigenvalues $\omega_i^2, (i = 1, 2)$ and natural modes $\boldsymbol{X}_i, (i = 1, 2)$ of this linear system as well. Normalize the natural modes as $\boldsymbol{X}_i = [1 \;\; X]^T$.

Question II. As shown in the right figure, a uniform bar of length r and mass m is placed on a semicircular surface of inner diameter r. A point mass of mass $2m$ is attached to a point A, one end of the bar, and the bar is in static equilibrium at a position inclined by θ from the horizontal as shown in the figure. Answer the following questions. The x-axis is horizontal and the y-axis is vertical. The semicircular surface is smooth and there is no friction between it and the bar. Let g be the acceleration of gravity.(25 points)



(1) Express the coordinates (x_G, y_G) of the midpoint G of the bar using θ.
(2) Express the coordinates (x_A, y_A) of the mass point using θ.
(3) Find the tangent of θ ($\tan\theta$) using the principle of the virtual work.