2025 ENTRANCE EXAMINATION FOR INTERNATIONAL MASTER'S PROGRAM

Departments of Mechanical Engineering and Hydrogen Energy Systems

Mathematics No.1 of 3 [13:15 $\sim$ 14:15]

Examinee's number ______________________

Score __________

I. Find the general solution for the following differential equations (30 points).

I-1) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2 - 4xy^3}{6x^2y^2 - y}$

I-2) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3y^3$

I-3) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x^2 + x + 1$

Answer below the line

Mathematics No.2 of 3 [13:15 $\sim$ 14:15]

Examinee's number ______________________

Score ________

II. Answer the following questions regarding the ellipsoid represented by:

$$x^2 + 4y^2 + z^2 = 16 \tag{1}$$

The objective here is to find the volume V of the ellipsoid in Cartesian coordinates (40 points).

II-1) Consider the function $f = x^2 + 4y^2 + z^2 - 16$. Let $\boldsymbol{r} = (x, y, z)$ be an arbitrary vector from the center of the ellipsoid to a point inside it. Find the function $\phi(x, y) = \boldsymbol{r} \cdot \boldsymbol{n}$, where $\boldsymbol{n} = \nabla f / |\nabla f|$ is the unit normal outward vector, and gradient operator ∇ is defined as:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{2}$$

II-2) Let D be the projection of the surface $z = z(x, y)$ onto the xy-plane. A symmetrical volume V is defined with the surface $z = z(x, y)$ as follows:

$$V = \frac{2}{3}\int_S \phi(x, y)\mathrm{d}S \tag{3}$$

where S is the portion of the surface with $z \geq 0$:

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y \tag{4}$$

Using equations (3) and (4), find the volume V of the ellipsoid.

Answer below the line

Mathematics No.3 of 3 [13:15 $\sim$ 14:15]

Examinee's number ____________________

Score ________

III. Let $\boldsymbol{A}$ be a constant wide matrix that is of row full rank, $\boldsymbol{M}$ be a constant positive definite matrix, and $\boldsymbol{x}$ and $\boldsymbol{b}$ be non-zero vectors. Find the vector $\boldsymbol{x}$ that minimizes the quadratic form $\frac{1}{2}\boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x}$ subject to the linear constraint $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ using the method of Lagrange multipliers. Here, let $\boldsymbol{\lambda}$ be the Lagrange multiplier vector and the Lagrangian function is defined as follows:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x} - \boldsymbol{\lambda}^\top \left(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\right)$$

(30 points)

Answer below the line